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A simple iterative approach for some fractional order models of engineering applications

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Abstract

The main topic of this paper is implementing an iterative approach based on the LA transformation (LAT) for solving fractional order-partial differential equations (FO-PDEs) offering valuable insights and practical solutions for a wide range of scientific and engineering applications. Several examples are presented, covering various physical and mathematical problems. The solution process is explained step-by-step, depicting how LAT can effectively handle fractional-order derivatives and achieve efficient approximated and analytical solutions. The Caputo operator is utilized to express the fractional-order derivatives. The paper explores various examples involving fractional diffusion equations, fractional Burger's equation, and fractional Navier-Stokes equation, among others. This method ensures convergence toward the exact solution for FO-PDEs and has been validated through the presentation of several examples that demonstrate its accuracy. This study contributes to the advancement of fractional calculus techniques and their utilization in real-world problem-solving scenarios.

Keywords: LA transformation, fractional-order partial differential equations, Integral transform methods, Navier-Stokes equation.

1. Introduction

FO-PDEs have gained increasing attention due to their ability to model complex systems accurately. In general, solving differential equations poses significant challenges, leading to the need for the creation of specialized mathematical techniques. Numerous analytical and numerical methods have been devised to solve them such as fractional calculus techniques[1], Analytical Adomian Decomposition Method (AADM) [2-7], Variational Iteration Method (VIM) [8-11], Laplace Decomposition Method (LDM) [12-16] and numerically we have Finite Difference[17] and Finite Element Methods, Spectral Methods such as Fourier series or Chebyshev polynomials, and Fractional Difference Methods.

Finally, Integral transform methods have proven to be extremely valuable tools for solving FO-PDEs [18-20]. These approaches use integral transforms, such as the Fourier [21-23], Laplace [24-26], Sumudu [27-29] or Mellin [30] [31] transforms, to convert the fractional-PDE into an ordinary differential equation (ODE) or algebraic equation.

Recently, several new classes of Laplace transforms have been discovered, including Elzaki [32-36], Natural [37], Aboodh [38], ZZ transform [39], Kamal transform [40] and the major transform of this research which is the LAT [41]. These techniques offer alternative approaches and methodologies for solving various mathematical challenges. Their discovery has expanded the range of tools available for tackling differential equations and other mathematical problems, giving researchers more options and flexibility in their analyses. It is worth emphasizing that each transform mentioned has its advantages and limitations, and the choice of method depends on the specific characteristics of the fractional differential equation and the desired accuracy of the solution.

Furthermore, In our approach, we employed fractional derivatives, such as the Caputo and Riemann-Liouville derivatives [42-44], have been developed specifically for fractional calculus and have proven useful in solving FO-PDEs. providing valuable tools for analyzing and obtaining solutions to both linear and nonlinear FO-PDEs, adding to the understanding and progress of this discipline and well-suited for problems that involve initial value conditions with fractional derivatives.

The LAT was first brought up by Luma and Alaa K. Jabber [41], introducing its fundamental properties and discussing all its applications. Additionally, it has been employed to tackle the solution of 2 and 1-dimensional partial differential equations (PDEs) by integrating it with a decomposition technique, as described in [45-49]. provides a systematic and analytical framework for obtaining the solution components. Also, researchers have successfully demonstrated and highlighted the application of the double LAT in solving diverse types of differential equations [50], and we believe that there are still undiscovered applications and potential uses for LAT that remain unexplored.

2. Fractional Calculus and transformation

In this section, we begin with introducing some fundamental definitions of fractional calculus as well as some properties of LAT strategy, which will be utilized in the following sections of this paper:

Definition (2.1): The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f(x) \in C_{\mu}, \mu \geq 1$; is,

$$J^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^t f(\tau) d\tau, \alpha > 0, x > 0 \\ J^0 f(x) = f(x), \alpha = 0 \end{cases} \quad (2.1)$$

where $\Gamma(..)$ is the known Gamma function as in [42-44].

Definition (2.2): The Caputo sense which expresses the fractional derivative of the function $f(x)$ is defined as[11,26],

$$ {}_0^cD_x^\alpha f(x) = \begin{cases} J^{n-\alpha}D^n f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, \\ \frac{d^n}{dt^n} f(x), at\ \alpha = n \end{cases} \quad (2.2) $$

where $n-1<\alpha<n, n\in N, x>0, f\in C_{-1}^n$.

Note the following two basic properties of D^α :

- $D^\alpha J^\alpha f(x) = f(x)$ (2.3)
- $J^\alpha D^\alpha f(x) = f(x) - \sum_{i=0}^{k-1} f^{(i)}(0^+)\frac{x^i}{i!}, x > 0$ (2.4)

Definition (2.3): The LAT of a function $f(x)$, denoted by $\bar{F}(u)$, is defined by the equation[41] .

$$ \bar{F}(u) = \mathrm{T}\{f(x)\} = \int_0^\infty e^{-x} f\left(\frac{x}{u}\right)dx, x>0. \quad (2.5) $$

Where u is a real number, at which the improper integral converges.

Its inverse can be expressed as a linear combination. as follows,

$$ T^{-1}\{\sum_{k=1}^n a_k \bar{F}_k(x,u)\} = \sum_{k=1}^n a_k\, T^{-1}\{\bar{F}_k(x,u)\} \quad (2.6) $$

Definition (2.4): by applying LAT on the fractional derivative $D^\alpha f(x)$ we get that form:

$$ T\{{}_0^cD_x^\alpha f(x), u\} = u^\alpha \bar{F}(u) - \sum_{k=0}^{n-1} u^{n-k} f^{(k)}(0), \quad (2.7) $$

Where $n-1<\alpha\le n, n\in N$.

3. Simple Iterative Approach

LAT of the convolution of the two functions $f(t)$and $g(t)$ which are equal to zero for $t<0$, Is equal to the product of the LAT of the two functions:

$$ f(t)*g(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau \quad (3.1) $$

$$ T\{f(t)*g(t),u\} = \frac{1}{u}\, T\{f(t)\}\, T\{g(t)\} \quad (3.2) $$

By using the formula of the LAT of the derivative of an integer order n of the function $f(t)$:

$$ T\{f^{(n)}(t),u\} = u^n\, T\{f\} - \sum_{k=0}^{n-1} u^{n-k} f^{(k)}(0) = u^n\, T\{f\} - \sum_{k=0}^{n-1} u^k f^{(n-k)}(0) \quad (3.3) $$

Now we shall apply LAT on fractional derivatives:

3.1. The Approach Implementation

As we have seen in Riemann-Liouville fractional integral definition, we can consider it as a convolution of two functions $f(t)$and $g(t) = t^{\alpha-1}$.

$$ D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau = t^{\alpha-1} * f(t). \quad (3.4) $$

By applying LAT on $g(t)$ is:

$$ \bar{G}(u) = T\{t^{\alpha-1}, u\} = \frac{\Gamma(\alpha)}{u^{\alpha-1}}, \quad (3.5) $$

Hence, by using the formula for the LAT of the convolution and here we obtained the LAT of the Riemann-Liouville fractional integral.

$$ T\{D_t^{-\alpha} f(t), u\} = \frac{1}{\Gamma(\alpha)}\, T\{f*g)\} = \frac{1}{\Gamma(\alpha)}.\frac{1}{u}.\bar{F}(u).\bar{G}(u) = u^{-\alpha}\bar{F}(u). \quad (3.6) $$

Now we are going to obtain a general formula for LAT handling the fractional derivatives by applying it to Caputo derivative eq (2.2).

rewriting the Caputo equation in the form:

$$ {}_0^cD_t^\alpha f(t) = D^{-(n-\alpha)} g(t), n-1<\alpha\le n \quad (3.7) $$

Assuming:

$$ g(t) = f^{(n)}(t), \quad (3.8) $$

By using eq (3.6) which describes the LAT of the Riemann-Liouville fractional integral we get

$$ T\{{}_0^cD_t^\alpha f(t), u\} = u^{-(n-\alpha)}\bar{G}(u) \quad (3.9) $$

By recalling the differential property of LAT on n^{th} Derivatives where n is any nonnegative integer,

$$ \begin{aligned} \bar{G}(u) &= T\{D_t^n f(t)\} \\ &= u^n\bar{F}(u) - \sum_{k=0}^{n-1} u^{n-k} f^{(k)}(0) \quad (3.10) \\ &= u^n\bar{F}(u) - \sum_{k=0}^{n-1} u^k f^{(n-k)}(0) \end{aligned} $$

So, by employing (3.10) into (3.9), we get the general formula of LAT for Caputo fractional derivative.

$$ T\{{}_0^cD_t^\alpha f(t), u\} = u^\alpha\bar{F}(u) - \sum_{k=0}^{n-1} u^{n-k} f^{(k)}(0), \quad (3.11) $$

Where $n-1<\alpha\le n$.

This section conveys the algorithm for solving the fractional differential equations using LAT, considering a general equation to be solved as:

$$ D_t^{m\alpha} f(x,t) + M[x]f(x,t) + N[x]f(x,t) = w(x,t), \quad (3.12) $$

$$ t>0, \quad x>0, \quad m-1<m\alpha<m $$

where $D_t^{m\alpha} = \frac{\partial^{m\alpha}}{\partial t^{m\alpha}}$ the Caputo fractional derivative in order $m\alpha$, M, N are the linear and nonlinear functions, w the source function.

The solution begins with applying LAT on both sides of eq (3.12),

$$ T\{D_t^{m\alpha} f(x,t)\} + T\{M[x]f(x,t) + N[x]f(x,t)\} = T\{w(x,t)\}, \quad (3.14) $$

using the differential property of LAT,

$$u^{m\alpha}\,\overline{F}(x,u)-\sum_{k=0}^{n-1}u^k\,D_t^{m\alpha-k}f(x,t)$$

$$=-T\{M[x]f(x,t)+N[x]f(x,t)\}+\,T\{w(x,t)\}, \quad (3.15)$$

so,

$$\overline{F}(x,u)=\frac{1}{u^{m\alpha}}\sum_{k=0}^{n-1}u^k\,D_t^{m\alpha-k}f(x,t)-\frac{1}{u^{m\alpha}}T\{M[x]f(x,t)+N[x]f(x,t)\}+\frac{1}{u^{m\alpha}}T\{w(x,t)\}, \quad (3.16)$$

Then the inverse of LAT is applied on eq (3.16),

$$f(x,t)=T^{-1}\left\{\frac{1}{u^{m\alpha}}\sum_{k=0}^{n-1}u^k\,D_t^{m\alpha-k}f(x,t)\right\}-T^{-1}\left\{\frac{1}{u^{m\alpha}}T\{M[x]f(x,t)+N[x]f(x,t)\}+\frac{1}{u^{m\alpha}}T\{w(x,t)\}\right\}, \quad (3.17)$$

We can say the general form of solution will be as:

$$f(x,t)=G_0(x,t)+G_1(x,t)-T^{-1}\left\{\frac{1}{u^{m\alpha}}T\{M[x]f(x,t)+N[x]f(x,t)\}\right\}, \quad (3.18)$$

Where $G_0(x,t), G_1(x,t)$ are the terms retrieved from the source function and the provided initial conditions where:

$$\mathrm{G}(x,t)=G_0(x,t)+G_1(x,t), \quad (3.19)$$

Now, the simple iterative equation from equation (3.18) is.

$$f_{n+1}(x,t)=f_0(x,t)-T^{-1}\left\{\frac{1}{u^{m\alpha}}T\{M[x]\,f_n+N[x]\,f_n\}\right\} \quad (3.20)$$

Where $f_0(x,t)=G(x,t)$, (3.21)

By assuming the approximate

$$f(x,t)=\sum_{i=0}^{n}f_i(x,t) \quad , (3.22)$$

Then the successive equations are:

$$f_1(x,t)=-T^{-1}\left\{\frac{1}{u^{m\alpha}}T\{M[x]f_0+N[x]f_0\}\right\},$$

$$f_2(x,t)=-T^{-1}\left\{\frac{1}{u^{m\alpha}}T\{M[x]f_1+N[x]f_1\}\right\}, \quad (3.23)$$

...

$$f_i(x,t)=-T^{-1}\left\{\frac{1}{u^{m\alpha}}T\{M[x]f_{i-1}+N[x]f_{i-1}\}\right\},$$

Finally, with the following iterations, we get an approximated solution which mostly depends on the choice of $G_0(x,t)$ and $G_1(x,t)$.

4. Applications and Case Studies

In this section, we shall use LAT combined with an iterative approach in solving linear and nonlinear fractional differential equations.

4.1 Application:(fractional differential equation)

considering the following linear fractional differential equation[32] to be solved as follows,

$$D_t^{2\alpha}f(t)+f(t)=0, f(0)=f'(0)=1. \quad (4.1)$$

First, we can write eq (4.2) in the following form,

$$D_t^{2\alpha}f(t)=-f(t) \quad (4.2)$$

Applying LAT on eq (4.2) we get:

$$T\{\,D_t^{2\alpha}f(t),u\}=\,u^{2\alpha}\,\overline{F}(u)-u^{2\alpha}f(0)-u^{2\alpha-1}f'(0)=-\,\overline{F}(u) \quad (4.3)$$

Using the given initial conditions in eq (4.1) we get,

$$T\{f(t)\}=\bar{F}(u)=\,1+\frac{1}{u}-\frac{1}{u^{2\alpha}}\,\overline{F}(u) \quad (4.4)$$

As,

$$f(t)=\sum_{n=0}^{\infty}f_n(t), \quad (4.5)$$

Now solving for $f(t)$ by applying inverse of LAT,

$$f(t)=\sum_{n=0}^{\infty}f_n(t)=T^{-1}\left(1+\frac{1}{u}\right)-T^{-1}\left(\frac{1}{u^{2\alpha}}\,\overline{F}(u)\right) \quad (4.6)$$

So, we have:

$$f_0(t)=1+t, \quad (4.7)$$

$$f_{n+1}(t)=-T^{-1}\left(\frac{1}{u^{2\alpha}}\,\overline{F}_n(u)\right), n=0,1,2,\ldots$$

Expanding the solution as the following series:

$$f_0(t)=1+t$$

$$f_1(t)=-T^{-1}\left(\frac{1}{u^{2\alpha}}\,\overline{F}_0(u)\right)=-\frac{t^{2\alpha}}{(2\alpha)!}-\frac{t^{2\alpha+1}}{(2\alpha+1)!}, \quad (4.8)$$

$$f_2(t)=\frac{t^{4\alpha}}{(4\alpha)!}+\frac{t^{4\alpha+1}}{(4\alpha+1)!},$$

...

And so, we get the solution: $f(t)=f_0(t)+f_1(t)+f_2(t)+\cdots$

when $\alpha=1$, the solution obviously converges to the exact solution of the eq. (4.1) which is as follows,

$$f(t)=1+t-\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}-\frac{t^6}{6!}-\frac{t^7}{7!}+\ldots\cong\,sin(t)+cos(t). \quad (4.9)$$

4.2 Application:

Solving the following ODE of fractional order α where $0 < \alpha \leq 1$,

$$D_t^{\alpha} f(t) + f(t) = 0, f(0) = 1, \quad (4.10)$$

and has the exact solution $f(t) = e^{-t}$ as in [32].

First applying LAT on eq (4.10) we get:

$$T\{ D_t^{\alpha} f(t), u\} = u^{\alpha}\,\overline{F}(u) - u^{\alpha} f(0) = -\overline{F}(u) \quad (4.11)$$

Using the initial condition in eq (4.11) we get,

$$T\{f(t)\} = 1 - \frac{1}{u^{\alpha}}\,\overline{F}(u) \quad (4.12)$$

Applying the inverse of LAT,

$$f(t) = \sum_{n=0}^{\infty} f_n(t) = T^{-1}\left(1 - \frac{1}{u^{\alpha}}\,\overline{F}_n(u)\right), \quad n=0,1,2,3, \quad \dots \quad (4.13)$$

Consequently,

$$f_0(t) = 1,$$

$$f_1(t) = -T^{-1}\left(\frac{1}{u^{\alpha}}\,\overline{F}_0(u)\right) = -\frac{t^{\alpha}}{\alpha!}, \quad (4.14)$$

$$f_2(t) = \frac{t^{2\alpha}}{(2\alpha)!},$$

...

Finally,

$$f(t) = 1 - \frac{t^{\alpha}}{\alpha!} + \frac{t^{2\alpha}}{(2\alpha)!} - \frac{t^{3\alpha}}{(3\alpha)!} + \cdots \cong e^{-t}, when\ \alpha = 1 (4.15)$$

4.3 Application:

Solving the linear fractional PDE problem[51]:

$$D_t^{\alpha} f(x,t) = \frac{1}{2}\left[\frac{\partial f}{\partial x} - f\right], f(x,0) = 6e^{-3x}, x, t > 0. \quad (4.16)$$

Applying LAT on eq (4.16),

$$u^{\alpha}\,\overline{F}(x,u) = u^{\alpha} f(x,0) + \frac{1}{2}\frac{\partial}{\partial x} T\big(f(x,t)\big) - \frac{1}{2} T(f(x,t)), \quad (4.17)$$

Using the initial condition value

$$\overline{F}(x,u) = 6e^{-3x} + \frac{1}{2u^{\alpha}}\frac{\partial}{\partial x} T\big(f(x,t)\big) - \frac{1}{2u^{\alpha}} T(f(x,t)), \quad (4.18)$$

Applying the inverse of LAT we get,

$$f(x,t) = \sum_{n=0}^{\infty} f_n(x,t) = T^{-1}(6e^{-3x}) + T^{-1}\left(\frac{1}{2u^{\alpha}}\frac{\partial}{\partial x} T\big(f(x,t)\big) - \frac{1}{2u^{\alpha}} T(f(x,t))\right), \quad (4.19)$$

Consequently,

$$f_0(t) = 6e^{-3x}, \quad (4.20)$$

$$f_1(t) = T^{-1}\left(\frac{1}{2u^{\alpha}}\frac{\partial}{\partial x} T(f_0) - \frac{1}{2u^{\alpha}} T(f_0)\right) = -\frac{12e^{-3x} t^{2\alpha-1}}{(2\alpha-1)!},$$

$$f_2(t) = \frac{24e^{-3x} t^{3\alpha-1}}{(3\alpha-1)!},$$

...

Also, when as $\alpha = 1$the approximated solution becomes,

$$f(x,t) = 6e^{-3x} - 12e^{-3x} t + 24e^{-3x}\frac{t^2}{2!} - 48e^{-3x}\frac{t^3}{3!} + 96e^{-3x}\frac{t^4}{4!} + \cdots, \quad (4.21)$$

Which converges to the exact solution $6e^{-3x-2t}$

4.4 Application:(Burger's equation)

Solving a linear one-dimensional fractional Burger's equation[9] [52].

$$D_t^{\alpha} f(x,t) = f_{xx} - f_x + \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2, \quad (4.22)$$

where $x, t \geq 0, 0 < \alpha \leq 1$.

With initial condition $f(x,t) = x^2$.

The solution begins with applying LAT on eq (4.22) as,

$$u^{\alpha}\,\overline{F}(x,u) = u^{\alpha} f(x,0) + T\left(\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\right) + T(f_{xx} - f_x), \quad (4.23)$$

Then solving for f by using the inverse of LAT we get,

$$f(x,t) = \sum_{n=0}^{\infty} f_n(x,t)$$

$$= x^2 + T^{-1}\left(\frac{1}{u^{\alpha}} T\left(\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\right)\right) + T^{-1}\left(\frac{1}{u^{\alpha}} T\big(f_{n_{xx}} - f_{n_x}\big)\right) \quad (4.24)$$

So,

$$f_0(t) = x^2 + \frac{t^2}{2!} + (2x-2)\frac{t^{\alpha}}{\alpha!},$$

$$f_1(t) = T^{-1}\left(\frac{1}{u^{\alpha}} T\big(f_{0_{xx}} - f_{0_x}\big)\right) = (2-2x)\frac{t^{\alpha}}{\alpha!} - 2\frac{t^{2\alpha}}{2\alpha!}, \quad (4.25)$$

$$f_2(t) = 2\frac{t^{2\alpha}}{2\alpha!},$$

...

If the noise terms have been removed and, as α converges to be 1 the approximated solution converges to the exact solution,

$$f(x,t) = x^2 + \frac{t^2}{2}. \quad (4.26)$$

4.5 Application:(Riccati equation)

Solving the following non-linear Riccati differential equation in fractional order α [53]as,

$$D_t^{\alpha} f(t) = 1 - t^2 + f^2(t), f(0) = 0, \quad (4.27)$$

$$0 < \alpha \le 1.$$

which has an exact solution $f(t) = t$,

following the same procedures as the previous applications,

Applying LAT on eq (4.27) we get:

$$u^{\alpha}\,\overline{F}(u) - u^{\alpha} f(0) = 1 - \frac{2}{u^2} + T\left(f^2(t)\right), \quad (4.28)$$

Substituting with initial condition and applying LAT inverse

$$f(t) = \sum_{n=0}^{\infty} f_n(t) = T^{-1}\left(\frac{1}{u^{\alpha}} - \frac{2}{u^{\alpha+2}}\right) + T^{-1}\left(\frac{1}{u^{\alpha}}\,T\left(f^2{}_{n-1}(t)\right)\right), \quad (4.29)$$

Then,

$$f_0(t) = \frac{t^{\alpha}}{\alpha!} - 2\frac{t^{\alpha+2}}{(\alpha+2)!}, \quad (4.30)$$

$$f_1(t) = T^{-1}\left(\frac{1}{u^{\alpha}}\,T\left(f^2{}_0(t)\right)\right) = \frac{t^{3\alpha}\times 2\alpha!}{3\alpha!\times(\alpha!)^2} - 4\frac{t^{3\alpha+2}\times(2\alpha+2)!}{\alpha!\times(\alpha+2)!\times(3\alpha+2)!} + 4\frac{t^{3\alpha+4}(2\alpha+4)!}{(3\alpha+4)!\times[(\alpha+2)!]^2},$$

...

Finally, as α approaches to 1 the approximated solution approaches to the exact one as follows,

$$f(t) = t - \frac{2t^5}{15} + \frac{2t^7}{63} - \frac{4t^9}{405} + \frac{134t^{11}}{51975} - \frac{4t^{13}}{12285} + \frac{t^{15}}{59535} + \cdots, \quad (4.31)$$

4.6 Application: (The logistic equation)

Solving the following application which is the logistic non-linear fractional differential equation[35], where $f(t)$ is the population size of the species at a time t.

$$D_t^{\alpha} f(t) = \frac{1}{4}[f(t) - f^2(t)], f(0) = \frac{1}{3}, 0 < \alpha \le 1. \quad (4.32)$$

Now by applying LAT on eq (4.32), we get,

$$u^{\alpha}\,\overline{F}(u) = \frac{u^{\alpha}}{3} + T\left(\frac{f(t)}{4} - \frac{f^2(t)}{4}\right), \quad (4.33)$$

$$\overline{F}(u) = \frac{1}{3} + \frac{1}{u^{\alpha}} T\left(\frac{f(t)}{4} - \frac{f^2(t)}{4}\right), \quad (4.34)$$

Then applying the inverse of LAT we obtain,

$$f(t) = \sum_{n=0}^{\infty} f_n(t) = T^{-1}\left(\frac{1}{3}\right) + T^{-1}\left(\frac{1}{u^{\alpha}} T\left(\frac{f_n(t)}{4} - \frac{f^2{}_n(t)}{4}\right)\right), \quad (4.35)$$

Where, $f_0(t) = \frac{1}{3}$,

$$f_1(t) = \frac{t^{2\alpha}}{72*2\alpha!} - \frac{t^{3\alpha}\times 2\alpha!}{1296\times 3\alpha!\times(\alpha!)^2} \quad (4.36)$$

...

Also, when as $\alpha = 1$the approximated solution becomes,

$$f(t) = \frac{1}{3} + \frac{t}{18} + \frac{t^2}{144} + + \frac{5t^3}{15552} - \frac{t^4}{62208} + \cdots \cong \frac{e^{0.25t}}{2+e^{0.25t}}\ (exact\ solu). \quad (4.37)$$

4.7 Application: (Laplace equation)

Solving Laplace equation, with initial conditions in an FO-PDE form[51]:

$$D_t^{2\alpha} f(x,t) = -\frac{\partial^2 f}{\partial x^2}, f(x,0) = 0,\ f_t(x,0) = cosx, x, t > 0. \quad (4.38)$$

By applying LAT on eq (4.38),

$$u^{2\alpha}\,\overline{F}(x,u) - u^{2\alpha} f(x,0) - u^{2\alpha-1} f_t(x,0) = -\frac{\partial^2}{\partial x^2}\,\overline{F}(x,u), \quad (4.39)$$

Using initial conditions and applying inverse of LAT Sequentially,

$$\overline{F}(x,u) = \frac{1}{u^{2\alpha}}\left(u^{2\alpha-1} cosx - \frac{\partial^2}{\partial x^2}\,\overline{F}(x,u)\right), \quad (4.40)$$

$$f(x,t) = \sum_{n=0}^{\infty} f_n(x,t) = T^{-1}\left(\frac{1}{u} cosx\right) - T^{-1}\left(\frac{1}{u^{2\alpha}}\frac{\partial^2}{\partial x^2}\,\overline{F}(x,u)\right), \quad (4.41)$$

Then we get the approximated solution terms

$$f_0(t) = t\ cosx\ ,$$

$$f_1(t) = -T^{-1}\left(\frac{1}{u^{2\alpha}}\frac{\partial^2}{\partial x^2}\,\overline{F}_0(x,u)\right) = \frac{t^{2\alpha+1}}{(2\alpha+1)!} cosx, \quad (4.42)$$

$$f_2(t) = \frac{t^{4\alpha+1}}{(4\alpha+1)!} cosx,$$

...

As α approaches to 1 the approximated solution polynomials approach to exact one,

$$f(x,t) = \cos x \left[t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots\right] \cong \cos x \sinh t \ \ (exact\ solu.), \tag{4.43}$$

4.8 Application:(telegraph equation)

Solving the telegraph equation [51] also in the form of an FO-PDE:

$$D_t^{2\alpha} f(x,t) = -2\sigma \frac{\partial f}{\partial t} + \sigma^2 \frac{\partial^2 f}{\partial x^2}, f(x,0) = cosx, f_t(x,0) = 0, x, t > 0. \tag{4.44}$$

Applying LAT we obtain,

$$u^{2\alpha}\,\overline{F}(x,u) - u^{2\alpha} f(x,0) - u^{2\alpha-1} f_t(x,0) = T(-2\sigma f_t + \sigma^2 f_{xx}), \tag{4.45}$$

Then LAT inverse,

$$f(x,t) = \sum_{n=0}^{\infty} f_n(x,t) = cosx + T^{-1}\left(\frac{1}{u^{2\alpha}} T\left(-2\sigma f_{n_t} + \sigma^2 f_{n_{xx}}\right)\right), \tag{4.46}$$

We get.

$$f_0(x,t) = cosx\ , \tag{4.47}$$

$$f_1(t) = -\sigma^2 cosx \frac{t^{2\alpha}}{(2\alpha)!},$$

$$f_2(t) = \sigma^3 cosx \left(2\frac{t^{4\alpha-1}}{(4\alpha-1)!} + \sigma \frac{t^{4\alpha}}{4\alpha!}\right),$$

...

As $\alpha = 1$ the summation of approximated solution parts approaches to

$$f(x,t) = (1 + 2\sigma) e^{-\sigma t} cosx. \tag{4.48}$$

4.9 Application:(diffusion equation)

Solving diffusion equation as a homogeneous non-linear fractional equation[54],

$$D_t^{\alpha} f(x,t) = f_{xx} - f_x + f f_{xx} - f^2 + f, f(x,0) = e^x, \quad 0 < \alpha < 1\,. \tag{4.49}$$

Applying LAT on eq (4.49),

$$\overline{F}(x,u) = f(x,0) + \frac{1}{u^\alpha} T(f_{xx} - f_x + f) + \frac{1}{u^\alpha} T(f f_{xx} - f^2), \tag{4.50}$$

Then inverse LAT consequently,

$$f(x,t) = \sum_{n=0}^{\infty} f_n(x,t) = e^x + T^{-1}\left(\frac{1}{u^\alpha} T\left(f_{n_{xx}} - f_{n_x} + f_n\right)\right) + T^{-1}\left(\frac{1}{u^\alpha} T\left(f_n f_{n_{xx}} - {f_n}^2\right)\right), \tag{4.51}$$

$$f_0(x,t) = e^x\ ,$$

$$f_1(t) = e^x \frac{t^\alpha}{\alpha!}, \tag{4.52}$$

$$f_2(t) = e^x \frac{t^{2\alpha}}{2\alpha!},$$

...

As $\alpha = 1$ the approximated solution polynomial approaches to exact one,

$$f(x,t) = e^x \left[1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \ldots\right] \cong \ (e^x e^t)(exact\ solu.), \tag{4.53}$$

4.10 Application: (Navier-Stokes equation)

Solving Navier-Stokes equation in cylindrical coordinates by considering some conditions of the flow of a one-dimensional, unsteady, viscous fluid in a tube[55]. The equations of motion which depict the previous conditions are given by.

$$D_t^{\alpha} f(x,t) = P + \frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\frac{\partial f}{\partial x}, 0 < \alpha \le 1, f(x,0) = 1 - x^2. \tag{4.54}$$

Where $\frac{\partial f}{\partial x} = f_x\,, \frac{\partial^2 f}{\partial x^2} = f_{xx}$

So, by applying LAT on eq (4.11) we get,

$$u^\alpha\,\overline{F}(x,u) - u^\alpha f(x,0) = T\left(P + f_{xx} + \frac{1}{x} f_x\right), \tag{4.55}$$

$$\overline{F}(x,u) = f(x,0) + \frac{P}{u^\alpha} + \frac{1}{u^\alpha} T\left(f_{xx} + \frac{1}{x} f_x\right), \tag{4.56}$$

Using the initial condition we get,

$$\overline{F}(x,u) = 1 - x^2 + \frac{P}{u^\alpha} + \frac{1}{u^\alpha} T\left(f_{xx} + \frac{1}{x} f_x\right) \tag{4.57}$$

Then applying inverse of LAT, we get:

$$f(x,t) = \sum_{n=0}^{\infty} f_n(x,t) = T^{-1}\left(1 - x^2 + \frac{P}{u^\alpha}\right) + T^{-1}\left(\frac{1}{u^\alpha} T\left(f_{xx} + \frac{1}{x} f_x\right)\right), \tag{4.58}$$

Consequently,

$$f_0(x,t) = 1 - x + P\frac{t^\alpha}{u^\alpha}$$

$$f_1(x,t) = T^{-1}\left(\frac{1}{u^\alpha} T\left(f_{0xx} + \frac{1}{x} f_{0x}\right)\right) = -4\frac{t^\alpha}{u^\alpha}, \tag{4.59}$$

$$f_2(t) = 0,$$

...

Here the approximate solution is obtained by summing all the previous terms in eq (4.59). This approximation closely resembles the exact solution of eq (4.54) when $\alpha = 1$ as,

$$f(x\ t) = 1 - x + (P - 4)\frac{t}{u}, \tag{4.60}$$

Conclusion

Overall, this paper demonstrates the effectiveness and applicability of the LAT in solving FO-PDEs. The presented mathematical models serve as valuable references for researchers and practitioners working in fractional calculus and its application fields. The insights gained from this study contribute to the advancement of fractional calculus techniques and their utilization in real-world problem-solving scenarios. We believe our approach can be a valuable tool for researchers and practitioners who are working on fractional differential equations.

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